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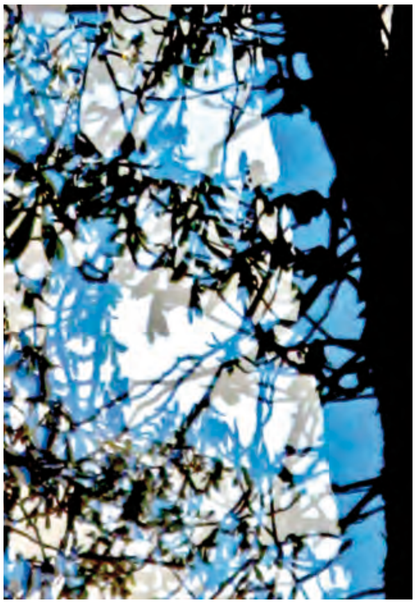
Synchronization



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Summary. In this paper we analyze the dynamics of two different models of oscillators. The most relevant aspect of both models is that synchronization emerges spontaneously as a natural stationary state and therefore may be a starting point for understanding complex patterns where exact timing plays a relevant role. However, the physical mechanisms leading to this temporal coherence are quite different in the two models, evidence of the richness of dynamic behavior in real systems. We are still far from a complete understanding of the whole process. The effect of the topology on the dynamics, the effect of mobility, the effect of disorder, etc., are all very important in biological, physical, and social environments and are the current focus of research in the field. [**Contrib Sci** 11(2): 207-214 (2015)]

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Introduction

A 2012 issue of *Scientific American* (October 2012 for the English version, December 2012 for the Spanish *Investigación y Ciencia*) contained an inspiring paper entitled "How cells communicate?" The article analyzed recent experiments that used multi-recording devices to simultaneously record the activity of several neurons. The novel technology and clever setup allowed the authors to perform measurements with a previously unattainable accuracy and thus provided very useful information about how we learn from experience. The main message was that precise timing plays a crucial role in certain perception tasks. One example is the auditory system, in which the arrival of a signal in just a few milliseconds is enough to allow discrimination between right and left and to determine the origin of the sound's source. To perform this task, cells (neurons) synchronize their activity via a remarkable process. The visual system is another illustrative example. Neurons capable of detecting features are distributed over different areas of the visual cortex. These neurons process

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information from a restricted region of the visual field and then integrate it through a complex, dynamic process that allows the detection of objects, their separation from the background, the identification of their characteristics, etc. Together, these tasks give rise to cognition. Experiments performed in the primary visual cortex of cats showed that some stimuli induce a correlation between the firing patterns of simultaneously recorded neurons, suggesting that certain global properties of stimuli can be identified through correlations in the temporal firing of different neurons. These oscillatory patterns may reflect the organized, temporally structured activity often associated with synchronous firing.

This phenomenology is not restricted to information processing in the brain. Synchronization is observed in biological, chemical, physical, and social systems and it has attracted the interest of scientists for centuries [9]. A paradigmatic example is the synchronous flashing of the fireflies found in some South Asian forests. At night, myriads of fireflies hover over the bushes. Suddenly, several of them start emitting flashes of light. Initially, they do so without any coherence, but after a short period of time the whole swarm is flashing in unison, creating one of the most striking visual effects ever seen. Another spectacular example concerns a group of metronomes. The reader can take advantage of the resources of the World Wide Web to watch several videos displaying the effect. Mechanical metronomes (those typically used in music) are placed on a table, each one with a random initial condition so that the global beat is incoherent. After a short transitory period (a couple of minutes) they adjust their relative phases so that they become temporally closer and closer, finally reaching a dynamic state in which the *orchestra* beats in unison in perfect synchrony. These are a few of the many examples from well-studied systems, but there are plenty of others in which the synchronized activity among members of a given population is the result of an emergent cooperative process.

Several questions arise immediately. When we consider concepts such as *timing* or *synchronization*, what are we talking about? Is it possible to precisely define the concept *synchronized state*? What is the physical mechanism that gives rise to it? These questions are by no means trivial and significant research efforts have been devoted to answering them. Arthur Winfree was one of the first scientists to seek an answer to the first question. He combined concepts from biology and mathematics to construct a theoretical framework in which formal ideas could be converted to mathematical modeling. His book "The Geometry of Biological Time" [11] is a summary of his seminal work. Winfree showed that there are many different ways to entrain two or more physical or biological entities. Phase locking, frequency locking, partial synchronization, total synchronization, m:n entrainment, etc., are different dynamic states characterized by a coherent temporal behavior among members of a population. Thus, synchronization is one of the most well studied emergent properties of complex systems, and it has remained so in different areas.

The mechanisms leading to these remarkable dynamic states can be analyzed from a physical perspective. There are two fundamental issues that deserve special attention. The first concerns the units themselves. They are usually considered as oscillators, either autonomous (keeping a natural rhythm on their own, such as pacemakers do), or exogenous (in which the oscillatory patterns are triggered by external stimuli). The second concerns the interaction between units, i.e., the mechanism by which they exchange information, which depends on the nature of the system under study. For instance, metronomes synchronize through a physical or mechanical mechanism (vibrations and movement of the air column), brain neurons do so through a combination of chemical and physical elements (such as electric currents plus synaptic neurotransmitters), whereas fireflies use flashes of light and visual communication. But we can also analyze synchronization in a more general way. Scientists working in this field tend to classify synchronization in two different major categories: diffusive and pulse-coupled systems. In the first case, the interaction between members of a population is considered a continuous time function, while the second (which concerns typically excitable systems) is characterized by a non-continuous, non-linear, time function, which makes the problem much more difficult to tackle, at least from a theoretical point of view. In this paper, we analyze two models extensively studied in the last years, each representing one of the aforementioned categories, and look at the precise mechanisms leading to synchronization. Although the stable attractor of the dynamics is the same (the synchronized state), the way it is reached substantially differs between the two models, thus providing useful insights on how information is processed in real systems. The two models are introduced below.

The Kuramoto model is the typical paradigm and the most extensively studied example of phase oscillators. Given a population of oscillators, it can be shown that, under certain conditions that affect the intensity of the coupling,

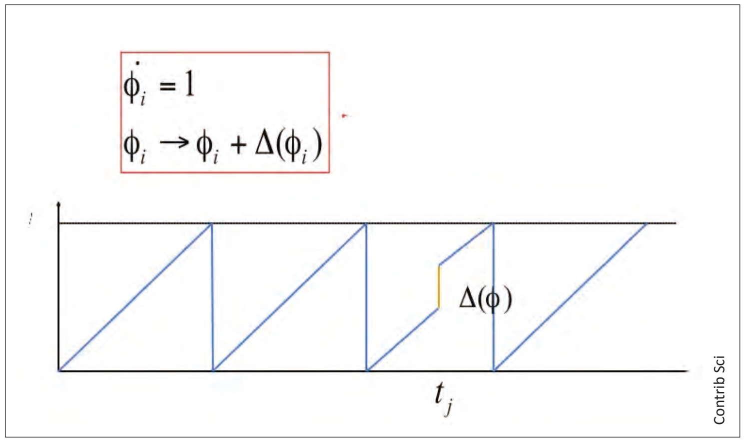

Fig. 1. Phase evolution of the oscillator's phase. When oscillator *j* fires at time t_j the phase of oscillator *i* advances by an amount that depends on its own phase.

the whole system can be treated as if only a globally limited cycle acts as an attractor of the dynamics. When this happens only one degree of freedom is needed to characterize the state of a given oscillator, its phase. Kuramoto [6] realized that just two elements are needed to obtain a non-trivial collective behavior. He assumed that every oscillator has a natural frequency, acquired from a random distribution; in the absence of coupling, each runs incoherently. In addition, he considered that the oscillators interact with each other through a non-linear function that depends on the phase difference between each pair. This type of coupling tends to synchronize the population. Therefore, in the complete model there is a tradeoff between two ingredients: if the distribution of frequencies is wide enough compared to the intensity of the coupling, there is no synchronization. In the opposite case, above a critical value of the coupling an emergent collective behavior arises and a fraction of the total population becomes synchronized (phase locked). A closer analysis of this model is presented later on.

The other system to be considered is an "integrate and fire" oscillator, which is a standard approach to excitable, pulse-coupled units. In the simplest description, it is assumed that a phase defined between [0,1] evolves in time with a constant velocity. When the phase reaches the upper value $\varphi = 1$, the unit fires, sending a signal to its neighbors, upon which it resets to $\varphi = 0$. When a unit receives the pulse, it changes its inner state according to a so-called phase response curve. The particular shape of this function depends on the system under consideration (for instance, it is quite different in cardiac vs. liver cells). We examine the dynamic behavior of this system in the following sections, starting with just two units and then extending the calculation to a population of N oscillators.

Two oscillators

To understand the emergence of synchronization in a set of dynamic systems, we first consider the simplest case of two oscillators [5]. Let us start with the "integrate and fire" oscillator model using two units whose phase evolves at a constant speed (equal to 1, without loss of generality):

$$\frac{d\varphi_{1,2}}{dt} = 1$$

The phase of each oscillator increases linearly in time until one of them reaches the threshold (assumed to be equal to 1) at which point the oscillator "fires," thus resetting its phase to 0 but sending a signal to the other oscillator that produces a sudden change in its phase (Fig. 1).

Schematically we can write the evolution of the phases of the oscillators as follows:

Osc 1	1	→	0	→	$1 - \varphi - \Delta(\varphi)$
		firing		driving	
Osc 2	φ	→	$\varphi + \Delta(\varphi)$	→	1

Initially, oscillator one has a phase equal to 1. After firing and resetting, followed by a driving, the phase of the second oscillator is equal to 1. This evolution allows us to identify a transformation:

$$\varphi \rightarrow 1 - \varphi - \Delta(\varphi)$$

We can then ask whether there exists a phase that represents a fixed point, i.e., a phase that is invariant after this transformation:

$$\varphi^* = 1 - \varphi^* - \Delta(\varphi^*)$$

Just two simple conditions are enough to ensure the

Stability

– $\Delta_1'(\phi) > 0$

$$\phi_0 = \phi^* + \delta \longrightarrow 1 - \phi^* - \delta - \underbrace{\Delta_1(\phi^* + \delta)}_{>\Delta_1(\phi^*)=1-2\phi^*} < \phi^* - \delta$$

$$\phi_0 = \phi^* - \delta \longrightarrow > \phi^* + \delta$$

– $\Delta_1'(\phi) < 0$

$$\phi_0 = \phi^* + \delta \longrightarrow > \phi^* - \delta$$

$$\phi_0 = \phi^* - \delta \longrightarrow < \phi^* + \delta$$

Both equations need another limit

$$\phi^* + \delta \quad \rightarrow \quad 1 - \phi^* - \delta - \underbrace{\Delta_1(\phi^* + \delta)}_{>1-2(\phi^*+\delta)} < \phi^* + \delta$$

$$\phi^* - \delta \quad \rightarrow \quad 1 - \phi^* + \delta - \Delta_1(\phi^* - \delta) > \phi^* - \delta$$

Fig. 2. Stability analysis of the fixed point.

existence and unicity of the fixed point of the transformation: (a) If $\Delta(\varphi)$ is bounded ($\Delta(\varphi) < 1$) and the function is continuous, then the fixed point exists; (b) If the derivative $\Delta'(\varphi) > -2$ for all values of the argument, then the fixed point is unique.

The existence of the fixed point is not sufficient to understand the collective behavior of the two oscillators. Instead, we must consider the stability of the fixed point; that is, rather than starting at the fixed point of the transformation, we start at a slightly different value, $\varphi = \varphi^* + \delta$. The behavior of this transformation is shown in Fig. 2. In the first case (positive derivative) the transformed phase is farther from the fixed point (on the opposite side) than the original phase. This means that the fixed point is unstable and acts as a repeller. On the other hand, after a half-cycle transformation, when the derivative is negative, the transformed phase is closer to the fixed point; hence the fixed point is an attractor.

Counterintuitively, the synchronization of the two oscillators lies in the fact that the fixed point is a repeller, such that after every transformation of the transformed phase is further and further from the fixed point, until it reaches the value 0 or 1 (from a periodic phase point of view, these two values are identical), in which case the two oscillators remain “synchronized” with exactly the same phase forever. However, if the fixed point was stable, after every transformation the phase becomes closer and closer to the fixed point, thus remaining out of phase with respect to the other oscillator.

In the synchronization of two pulse-coupled oscillators, the synchronization mechanism comes from the instability of the fixed point. This is, however, not always necessary; for instance, when the two oscillators are described by continuous equations in time, as happens in the Kuramoto model:

$$\frac{d\varphi_i}{dt} = \omega_i + K \sin\left(\varphi_j - \varphi_i\right) \qquad i, j = 1, 2; j \neq i$$

In this case there are two terms, the first corresponds to the natural frequency of every oscillator, and the second is the coupling term. When $K = 0$, the two oscillators are independent and will never synchronize. However, for increasing values of K there is some amount of phase entrainment between the two units, as can be easily deduced.

We now introduce two auxiliary variables $(x = \varphi_1 - \varphi_2, y = \varphi_1 + \varphi_2)$, for which the equations of motion are:

$$\frac{dx}{dt} = \omega_1 - \omega_2 - K \sin x$$

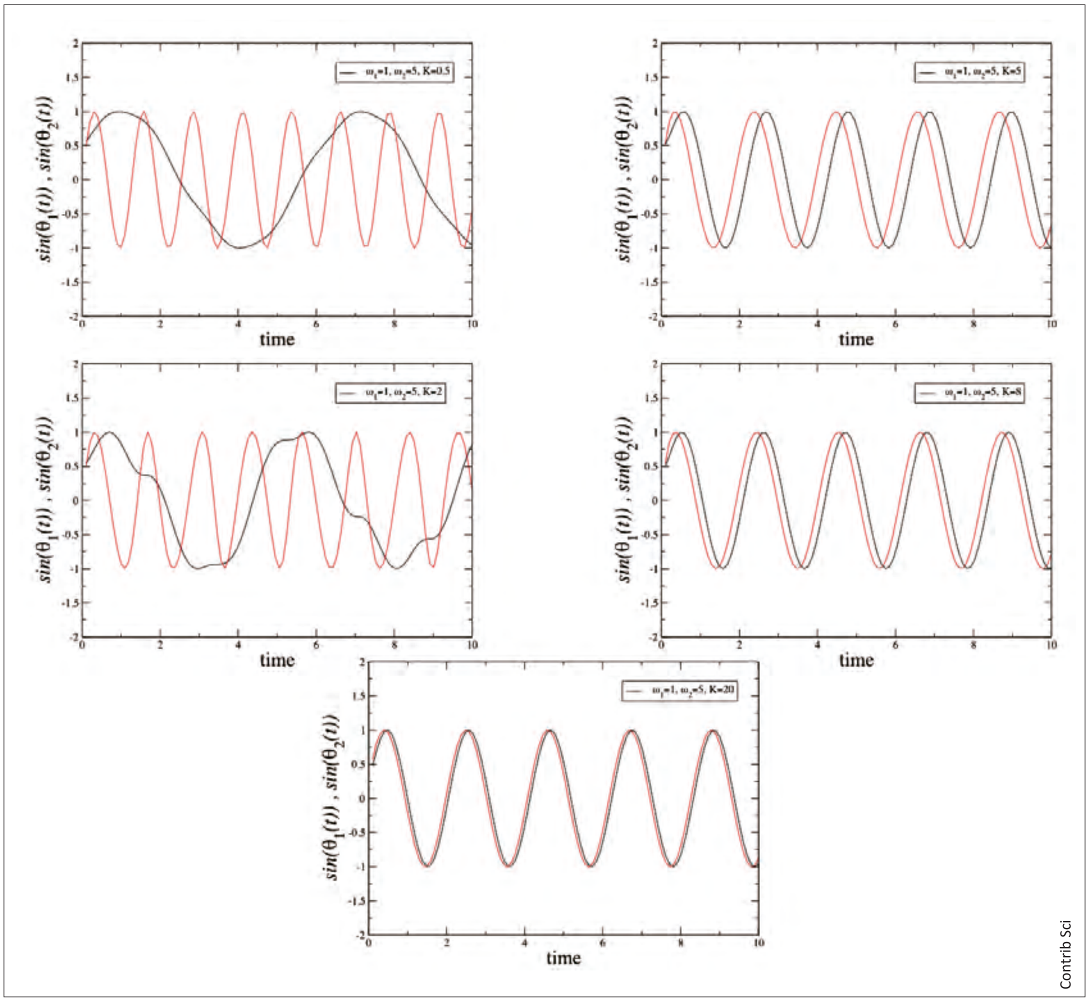

Fig. 3. Phase evolution for two oscillators coupled according to the Kuramoto model, for different values of the coupling constant.

$$\frac{dy}{dt} = \omega_1 + \omega_2$$

and from which we can conclude that when $K < |\omega_1 - \omega_2|$, the term on the right hand side is always positive or negative, meaning that *x* is an ever growing or ever decreasing function of time; therefore, the phases of the oscillators will tend to be apart. However, when $K > |\omega_1 - \omega_2|$ a steady state is reached, such that:

$$(\varphi_1 - \varphi_2) \underset{t\to\infty}{\rightarrow} \sin^{-1}\left(\frac{\omega_1 - \omega_2}{K}\right),$$

meaning that the phase difference tends to a constant value in time, i.e., the two oscillators will become entrained (Fig. 3). It is worth noting that when considering synchronization two situations must be distinguished: (a) strong synchronization, when all units have identical phases and frequencies; (b) weak synchronization (also known as phase entrainment), when the frequencies are identical, but the phases keep a constant difference in time

Having understood this simple setting, let us now analyze a globally coupled system.

N oscillators with global coupling

We start by considering the original Kuramoto model [1,6], in which the phase of each oscillator evolves in time according to:

$$\frac{d\varphi_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\varphi_j - \varphi_i\right)$$

$$i = 1, \ldots, N$$

where ω_i denotes the natural frequency picked up from a given probability distribution $g(\omega)$. In this model, because all the units are mutually interconnected, the factor K/N ensures a proper scaling behavior within the thermodynamic limit (large N). The basic concepts introduced in the previous section are still valid: there is a relationship between K and the frequency distribution that determines the appearance of a transition from an initially incoherent state to another, partially synchronized one. However, because finding this particular value is much more complicated, we need to resort to statistical mechanics, a discipline of physics able to deal with large populations.

The first step is to define an appropriate order parameter. Landau [7] proposed this approach to characterize phase transitions. It is known that symmetry breaking underlies a phase transition and the order parameter helps to identify when it occurs, since in one phase the order parameter is 0 while in the other it takes a non-vanishing value. For the Kuramoto model, the order parameter takes the form:

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\varphi_j}$$

where $r(t)$ with $0 < r(t) < 1$ measures the coherence of the oscillator population, and $\psi(t)$ is the average phase. With this definition, the dynamic equation becomes:

$$\frac{d\varphi_i}{dt} = \omega_i + K\,r\,\sin(\psi - \varphi_i) \qquad i = 1,\dots,N$$

In the limit of infinitely many oscillators, they will be distributed with a probability density $\rho(\varphi,\omega,t)$, such that:

$$re^{i\psi} = \int_{-\pi}^{\pi}\int_{-\infty}^{+\infty} e^{i\varphi}\rho(\varphi,\omega,t)\,g(\omega)\,d\varphi\,d\omega$$

An appropriate mathematical treatment of this probability density leads to:

$$r = Kr\int_{-\pi/2}^{\pi/2}(\cos\varphi)^2\,g(Kr\sin\varphi)\,d\varphi$$

This equation always has the trivial solution $r = 0$, corresponding to incoherence, $\rho = 1/(2\pi)$, which means that the phase of the oscillators is uniformly distributed over the circle. However, it also has a second branch of solutions, corresponding to the partially synchronized phase and satisfying:

$$1 = K\int_{-\pi/2}^{\pi/2}(\cos\varphi)^2\,g(Kr\sin\varphi)\,d\varphi$$

This branch bifurcates continuously from $r = 0$ at the value $K = K_c$, obtained by setting $r = 0$, which yields $K_c = 2/[\pi g(0)]$, where $g(0)$ is simply the distribution of frequencies evaluated at 0. Kuramoto was the first to devise this formula and the argument leading to it.

Regarding a population of N "integrate and fire" oscillators, the analytical procedure to elucidate the attractor of the dynamics is quite involved, but the basic idea is simple, and, again, we can use the previous example of two oscillators. The key element is to realize that when two units fire simultaneously, they keep firing in unison forever. In mathematical terms, this is called an absorption, which technically is equivalent to considering that the number of independent oscillating units is reduced. It can be shown that the probability of finding two not-absorbed units when $t \to \infty$ tends to zero, confirming that the final attractor of the dynamics is the fully synchronized state [8].

New paradigms

In the previous sections, we focused on two special limits, one in which connectivity is minimal and the other in which it is maximal. The former is a good example of simple mathematics that can be solved exactly, helping us to understand the emergence of certain collective behaviors. The latter is complicated from a mathematical point of view, but the fact that all units are connected allows certain approximations. The first step to obtaining more realistic structures is to consider regular settings, for instance, rings in one dimension, planes in two dimensions, and, in general, hypercubic lattices in any dimension. In this case, the previous approximations are not valid and new theories are needed. From a purely phenomenological point of view, the main findings include the potential appearance of new structures. Synchronization is indeed possible under certain circumstances, but other phenomena, such as phase-locking (in which effective frequencies are identical but phases are not), emerge. For instance, in pulse-coupled oscillators, when the fixed points are repellers, synchronization emerges, as is the case in reduced and all-to-all connectivities; but if the fixed points are attractors, different local structures are possible.

Nature and society are similarly organized, forming structures that are far from regularly connected and thus unlike those described previously. This also affects how complex systems can become synchronized. In a recent review [2], we presented a detailed overview of the different aspects that complex topologies represent for synchronization. Here, we summarize the main implications for the emergence of synchronization.

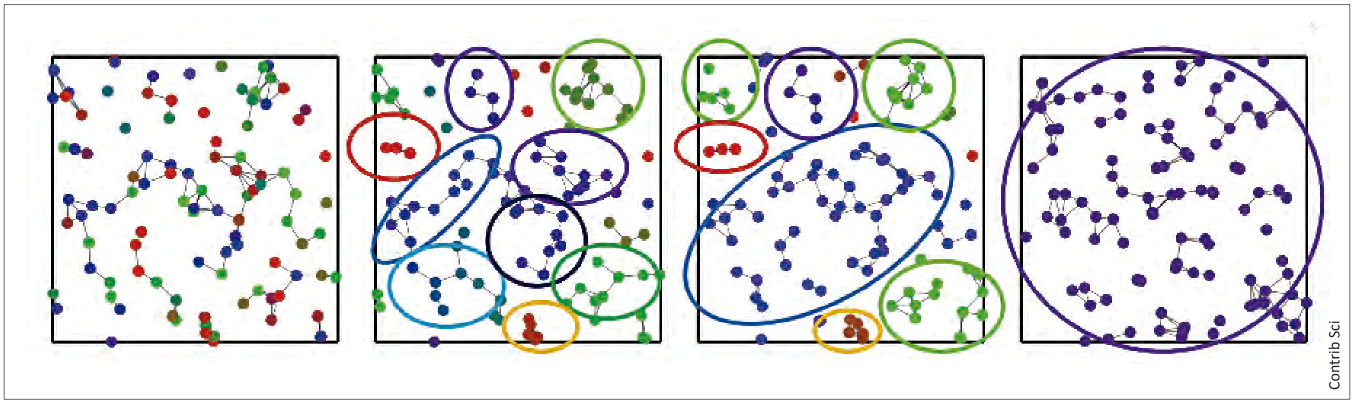

Fig. 4. The left to right movement over time of Kuramoto identical oscillators along a square. The colors correspond to phases in the interval [0,2π]. The Venn diagrams group oscillators that are already synchronized.

The modern theory of complex networks is intimately intertwined with synchronization. In their seminal paper, published in 1998 [10], Watts and Strogatz introduced a simple model structure (the small world network). This structure was originally considered as a necessary ingredient in the problem of synchronization of cricket chirps, since they show a higher than expected degree of synchronization with the chirps of distant peers, as if they were "connected." Unlike other models, the Watts and Strogatz model uses this as an initial setup and adds long-range random links between units, which makes the effective distances between units decrease substantially. This is but one of the many cases in science in which a proposed model makes a remarkable contribution to a field other than the one it was originally developed for.

In their paper, Watts and Strogatz noted a number of systems in which the connectivity patterns could be mapped using their model, showing, simultaneously, the effect of reducing the average distance between nodes (as they appear in random graphs) but also of keeping the local degree of clustering. And what is the effect of these new models of synchronization? Many researchers have turned their attention to the features of synchronization. Qualitatively, it can be stated that, indeed, the decrease in the average distance makes the units interact more strongly, thus enhancing synchronization. On the other hand, local irregularities prevent the emergence of certain heterogeneous structures.

The paper by Watts and Strogatz was followed shortly thereafter by a seminal paper on complex network science authored by Barabasi and Albert [4], which recognized that some real-world networks are even more "complex." The newly recognized feature was that the distribution of degrees was not close to the classical one, and a clear power-law decay with no characteristic scales was demonstrated. The implication of these so-called scale-free networks was that there are many nodes with a small number of connections, but some of them are highly connected, forming hubs. In this case, the focus on synchronization was directed to the role played by the different types of nodes, classified according to their topological properties, such as the degree or the different types of centrality.

It is clear, however, that there must be an intrinsic relation between topological scales and the dynamic evolution of the synchronization process. We previously showed that a system composed of identical Kuramoto oscillators evolving from random initial conditions towards the only attractor, the synchronized state, produces phase correlations (which act as a kind of local-order parameter) that are the dynamic consequence of the topological distribution of the network [3].

Last, but not least, an additional degree of complexity arises for networks that, having their own dynamic rules, evolve with time. This time dependence can have different origins; one that is quite easy to understand corresponds to the motions of the units. Thus, when units move very fast synchronization is enhanced (Fig. 4), whereas when their motion is very slow, they also reach the final synchronized state, but over a much longer scale and with a different mechanism. For intermediate velocities, the compromise between the two mechanisms can produce undesirable (or desirable) consequences and disable synchronization.

Competing interests. None declared.

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